Greedy Spanners in Euclidean Spaces Admit Sublinear Separators

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A $t$-spanner of a graph $G = (V, E, w)$ is a subgraph $H = (V, E_H, w)$ such that for every pair $(u, v) \in V^2$:

$$d_G(u, v) \leq d_H(u, v) \leq t \cdot d_G(u, v)$$

Parameter $t$ is called the **stretch** of the spanner.
Spanner

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Given a set of point $P$ in $\mathbb{R}^d$. In our paper, we focus on the spanner of the graph $G = (P, (P^2), \| \cdot \|_2)$. 
Applications

Complexity of Network Synchronization

BARUCH Awerbuch

Massachusetts Institute of Technology, Cambridge, Massachusetts

(a) Distributed Computing

EXPLORING PROTEIN FOLDING TRAJECTORIES USING GEOMETRIC SPANNERS

D. RusseL and L. Guibas

(b) Approximation Algorithm

Near Optimal Multicriteria Spanner Constructions in Wireless Ad-Hoc Networks

Hanan Shpungin, Member, IEEE, and Michael Segal, Senior Member, IEEE

(c) Computational Biology

Figure: Applications of spanners

(d) Wireless Sensor Network
A (balanced) separator $S$ is a subset of the vertex set of the graph $G = (V, E)$ such that each connected component of $G[V/S]$ has at most $\frac{2}{3}n$ vertices.

**Figure:** A separator $S$ of $G$
Greedy Spanner

**Algorithm** Greedy\((G = (V, E, w), t)\)

1: sort edges in \(E\) in increasing order
2: \(H = (V, \emptyset, w)\)
3: for \(e = (u, v) \in E\) in sorted order do
4:  if \(d_H(u, v) > t \cdot w(u, v)\) then
5:     \(E_H = E_H \cup \{e\}\)
6:  end if
7: end for
8: return \(H\)
Abam and Har-Peled (2010) constructed a \((1 + \epsilon)\)-spanner with a separator of size \(O(n^{1 - 1/d})\) for point set in Euclidean space with maximum degree \(O(\log^2 n)\).

**Open question**: Constructing a spanner with a sublinear separator and a constant maximum degree in metrics of constant doubling dimensions.
Results on Separators of Spanners

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  - **Open question**: Constructing a spanner with a sublinear separator and a constant maximum degree in metrics of constant doubling dimensions.

- Recently, Eppstein and Khodabandeh (2021) showed that the greedy spanner for point sets in \(\mathbb{R}^2\) admits a separator of size \(O(\sqrt{n})\).
  - **Open question**: Do greedy spanners for point sets in \(\mathbb{R}^d\) admit separator of size \(O(n^{1-1/d})\)?
Results on Separators of Spanners

- Abam and Har-Peled (2010) constructed a $(1 + \epsilon)$-spanner with a separator of size $O(n^{1-1/d})$ for point set in Euclidean space with maximum degree $O(\log^2 n)$.
  - **Open question**: Constructing a spanner with a sublinear separator and a constant maximum degree in metrics of constant doubling dimensions.
- Recently, Eppstein and Khodabandeh (2021) showed that the greedy spanner for point sets in $\mathbb{R}^2$ admits a separator of size $O(\sqrt{n})$.
  - **Open question**: Do greedy spanners for point sets in $\mathbb{R}^d$ admit separator of size $O(n^{1-1/d})$?

...
Contributions

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**Definition (Informal)**

A graph $G = (V, E)$ in a $\mathbb{R}^d$ is $\tau$-lanky if any ball $B(x, r)$ of radius $r$ is cut by at most $\tau$ edges of length at least $r$. 
Contributions

- Introduce $\tau$-lanky - a criterion of graphs admitting sublinear separators and bounded degree.
- $\tau$-lanky implies bounded degree and sublinear separator in strong sense, i.e. every subgraph admits sublinear separator.
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Theorem

Let $G = (V, E)$ be an $n$-vertex graph in $\mathbb{R}^d$ such that $G$ is $\tau$-lanky. Then, $G$ has a balanced separator $S$ such that $|S| = O(\tau n^{1-1/d})$ when $d \geq 2$ and the maximum degree of $G$ is $\tau$. 
Contributions

- Introduce $\tau$-lanky - a criterion of graphs admitting sublinear separators and bounded degree.

- $\tau$-lanky implies bounded degree and sublinear separator in strong sense, i.e. every subgraph admits sublinear separator.

- The criterion is simple, can be applied to non-spanner graphs.
Contributions

- Greedy spanner in $\mathbb{R}^d$ is $O(\epsilon^{1-2d})$-lanky.

- Greedy spanner has separator of size $O(n^{1-1/d})$. This resolves the open question in Eppstein and Khodabandeh (2021).

- Prove that a spanner in Chan et al. (2016) is $\epsilon - O(d)$-lanky, resolve the open question in Abam and Har-Peled (2010).
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- Prove that a spanner in Chan et al. (2016) is $\epsilon^{-O(d)}$-lanky, resolve the open question in Abam and Har-Peled (2010).
Outline

τ – lanky

Spanners in Euclidean Spaces

Spanners in Doubling Metrics

Conclusion
A graph $G = (V, E)$ in $\mathbb{R}^d$ is $\tau$-lanky if for any non-negative $r$, and for any ball $B(x, r)$ of radius $r$ centered at a vertex $x \in V$, there are at most $\tau$ edges of length at least $r$ that are cut by $B(x, r)$. 
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Figure: A ball with radius $r$ is cut by 8 edges with length at least $r$
Definition

A graph $G = (V, E)$ in $\mathbb{R}^d$ is $\tau$-lanky if for any non-negative $r$, and for any ball $B(x, r)$ of radius $r$ centered at a vertex $x \in V$, there are at most $\tau$ edges of length at least $r$ that are cut by $B(x, r)$.

There is nothing special about the length $r$. In fact, any length $cr$ with $c$ is a positive constant does the trick.
A $\tau$-lanky graph $G$ has maximum degree $\tau$.

Prove by looking at a ball $(u, d_{\text{min}}/2)$ for every vertex $u$ of $G$. 
Implications

**Theorem**

Let $G$ be an $n$-vertex graph in $\mathbb{R}^d$ such that $G$ is $\tau$-lanky. Then, $G$ has a balanced separator $S$ such that $|S| = O(\tau n^{1-1/d})$ when $d \geq 2$ and $|S| = O(\tau \log n)$ when $d = 1$. Furthermore, $S$ can be found in $O(\tau n)$ expected time.
Proof Sketch

The separator is the set of vertices inside $B(v, r^*)$ that incicdent to any edge cutting $B(v, r^*)$. 
Proof Sketch

- Find a smallest ball $B(v, r)$ that contains $\frac{n}{2d+1}$ vertices. Hence, $B(v, 2r)$ contains at most $\frac{n}{2}$ vertices.
Proof Sketch

- Choose a random radius $r^*$ uniformly in $[r, 2r]$. The expected number of edges with length $rn^{-1/d}$ cutting $B(v, r^*)$ is $O(n^{1-1/d})$. 

![Diagram of a random radius and its effect on the expected number of edges cutting a ball centered at v.](image)
Proof Sketch

- For edges with length in \([rn^{-1/d}, 2r]\), partition \(B(v, 2r)\) into smaller balls accordingly. Each edge cutting \(B(v, r^*)\) must also cut one small ball.

\[ \# \text{ edges cut } B(v, r^*) \leq \tau \times (\# \text{ balls}). \]

- Number of edges with length larger than \(2r\) cutting \(B(v, r^*)\) \(\leq \tau\).
Proof Sketch

- The separator is the set of vertices inside $B(v, r^*)$ that incident to any edge cutting $B(v, r^*)$. 
Algorithmic Implications

- Unweighted optimization problems such as independent set, vertex cover, dominating set, connected dominating set, packing problems, admit a polynomial-time approximation scheme (PTAS) in a graph has polynomial expansion.
- If the $G$ has bounded degree, then the vertex-weighted version of those problems admit a PTAS.
Every subgraph has sublinear separator

Graph has polynomial expansion

+ Bounded Degree

PTAS for unweighted Independent Set, Vertex Cover, Dominating Set, ...

PTAS for weighted Independent Set, Vertex Cover, Dominating Set, ...

Algorithmic Implications
Euclidean Spaces

**Theorem**

*Given a set of points $P$ in $\mathbb{R}^d$. Then the greedy spanner $G$ of $P$ is $\epsilon^{1-2d}$-lanky.*
Euclidean Spaces

Theorem

Given a set of points \( P \) in \( \mathbb{R}^d \). Then the greedy spanner \( G \) of \( P \) is \( \epsilon^{1−2d} \)-lanky.

Proof sketch:
Euclidean Spaces

Theorem

Given a set of points $P$ in $\mathbb{R}^d$. Then the greedy spanner $G$ of $P$ is $\varepsilon^{1-2d}$-lanky.

Proof sketch:

- For any two subsets $X$ and $Y$ of $P$ such that $\text{dist}(X, Y) \geq \frac{12}{\varepsilon} \text{diam}(X)$, there are $O(\varepsilon^{1-d})$ edges in $G$ between $X$ and $Y$. 
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Given a set of point $P$ in $\mathbb{R}^d$. Then the greedy spanner $G$ of $P$ is $\epsilon^{1-2d}$-lanky.

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- For any ball $B$ of radius $r$, partition $B$ into smaller balls of radius $\epsilon r / 48$, then each ball is cut by at most $O(\epsilon^{1-d})$ edges.
Extended Results

Theorem (Bounded fractal dimension)

Let $P$ be a given set of $n$ points in $\mathbb{R}^d$ that has fractal dimension $d_f$, and $G$ be the greedy $(1 + \epsilon)$-spanner of $P$. Then $G$ has a separator $S$ of size $O(n^{1-1/d_f})$. 

Theorem (Unit ball graphs)

Let $G$ be the greedy $(1 + \epsilon)$-spanner of a unit ball graph in $\mathbb{R}^d$. Then $G$ has a separator $S$ of size $O(|V(G)|^{1-1/d})$. 

Theorem (Bounded fractal dimension)

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Theorem (Unit ball graphs)

Let $G$ be the greedy $(1 + \epsilon)$-spanner of a unit ball graph in $\mathbb{R}^d$. Then $G$ has a separator $S$ of size $O(|V(G)|^{1-1/d})$. 
Lankiness in Metric Spaces

- Lankiness in Metric Spaces

- τ-lankiness's implications can be extended for other metric spaces.

- Definition

A metric \((X, \delta)\) is \((\eta, d)-\text{packable}\) if for any ball with radius 1, there are at most \(\eta\) points inside such that the distance between any two points is at least \(r\).
Lankiness in Metric Spaces

\( \tau \)-lanky’s implications can be extended for other metric spaces.
Lankiness in Metric Spaces

$\tau$-lanky’s implications can be extended for other metric spaces.

**Definition**

A metric $(X, \delta)$ is $(\eta, d)$-packable if for any ball with radius 1, there are at most $\frac{\eta}{rd}$ points inside such that the distance between any two points is at least $r$. 
Lankiness in Metric Spaces

\( \tau \)-lanky’s implications can be extended for other metric spaces.

**Definition**

A metric \((X, \delta)\) is \((\eta, d)\)-packable if for any ball with radius 1, there are at most \(\frac{\eta}{rd}\) points inside such that the distance between any two points is at least \(r\).

The Euclidean Space \(\mathbb{R}^d\) is \((\eta, d)\) packable with a constant \(\eta\). Moreover, the metric space with bounded doubling dimension \(d\) is also \((2^{O(d)}, d)\) packable.
Theorem

Let $G = (V, E, \delta)$ be an $n$-vertex graph in an $(\eta, d)$-packable metric $(X, \delta)$ such that $G$ is $\tau$-lanky. Then, $G$ has a balanced separator $S$ such that $|S| = O(\tau n^{1-1/d})$ when $d \geq 2$ and the maximum degree of $G$ is $\tau$. 
Chan, Gupta, Maggs and Zhou (CGMZ) constructed a $(1 + \epsilon)$-spanner with a maximum degree of $\epsilon^{-O(d)}$ for points in doubling metrics of dimension $d$.

- We prove their spanner is $\epsilon^{-O(d)}$-lanky.
Construction

- Construct a net tree $N$. In each level $i$, put all edges with length at most $(4 + \frac{32}{\varepsilon})r_i$ to the spanner.

![Net tree spanner diagram with levels and edges labeled]
Construction

- Reroute some edges of the spanner such that each edge connects vertices whose levels differ $1/\epsilon$.

Figure: Net tree spanner after rerouting
Spanners in Doubling Metrics

**Theorem**

CGMZ spanner $G$ of a set $P$ in doubling metrics of dimension $d$ is $\epsilon^{-O(d)}$-lanky.

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Proof sketch:

1. For any $\beta$-separated sets, there are at most $\left(\frac{4+32}{\epsilon}\beta\right)^2$ edges in $G$ between them.
2. Given a ball $B(p, r)$, there are $\epsilon^{-O(d)}$ edges with length $\left[\frac{4+32}{\epsilon}r, 4+32\right]$ cutting $B(p, r)$. This is proven by partitioning $B(p, r)$ into smaller balls.
3. For the edges with length more than $\left(\frac{4+32}{\epsilon}\right)r$, there are $\epsilon^{-O(d)}$ endpoints of those edges inside $B(p, r)$, which implies there are $\epsilon^{-O(d)}$ edges cutting $B(p, r)$ by the bounded degree.
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- For any $\beta$-separated sets, there are at most $\left(\frac{4+32/\epsilon}{\beta}\right)^{2d} \epsilon^{-O(d)}$ edges in $G$ between them.
- Given a ball $B(p, r)$, there are $\epsilon^{-O(d)}$ edges with length $[r, (4 + 32/\epsilon)r]$ cutting $B(p, r)$. This is proven by partitioning $B(p, r)$ into smaller balls.
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- For the edges with length more than $(4 + 32/\epsilon)r$, there are $\epsilon^{-O(d)}$ endpoints of those edges inside $B(p, r) \implies$ there are $\epsilon^{-O(d)}$ edges cutting $B(p, r)$ by the bounded degree.
Every $\tau$-lanky graph admits bounded maximum degree and sublinear separator (in strong sense).

Greedy spanner in $\mathbb{R}^d$ is $\epsilon$-1-2-d-lanky.

The bounded degree spanner in Chan et al. (2016) is $\epsilon-O(d)$-lanky.

Using the same technique, we proved that the greedy spanner in doubling metric admits separator of size $O(n^{1-1/d+\log(\text{spread})})$ with $\text{spread} = d_{\max}/d_{\min}$. 
Conclusion

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- The bounded degree spanner in Chan et al. (2016) is $\epsilon^{-O(d)}$-lanky.
- Using the same technique, we proved that the greedy spanner in doubling metric admit separator of size $O(n^{1-1/d} + \log(\text{spread}))$ with $\text{spread} = \frac{d_{\max}}{d_{\min}}$. 
Open question

Do greedy spanners admit sublinear separators (not depend on \textit{spread}) in doubling metrics?


Thank you for listening!
Question?