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Estimating Eigenvalues of Symmetric Matrices via Random Submatrices

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Problem Description

- **Given**: A **symmetric** matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ in the **bounded entry model** *i.e* $\|\mathbf{A}\|_\infty \leq 1$ [1].
- **Exact Eigenvalues**: SVD, power methods, etc. require reading the **full matrix** and have time complexity close to $O(n^\omega)$.
- **Faster** methods available for **PSD matrices**.
- **A** can be **indefinite** (**non-PSD**).
- **Problem**: Estimate **eigenvalues** of **A** upto ϵn **additive** error without using the full matrix.
- **Applications:** optimization, dynamical systems, and spectral graph theory.

Algorithm: Sampling Random Submatrices

- For each $i \in [1, n]$: **sample** i w.p. $\frac{s}{n}$: Sampled **Set S**.
- Get **principal submatrix** $\mathbf{A}_S$ corresponding to indices in **S**.
- Calculate eigenvalues of $\mathbf{A}_S$ and scale by $\frac{n}{s}$.

Theorem 1 *(Upper bound)*

For any $\lambda_i(\mathbf{A})$, such that $|\lambda_i(\mathbf{A})| \geq \epsilon\sqrt{\delta}n$, if $s \geq \tilde{O}(\frac{1}{\epsilon^3\delta})$, with probability at least $1-\delta$, we have,

$$\lambda_i(\mathbf{A}) - \epsilon n \leq \frac{n}{s}\lambda_i(\mathbf{A}_S) \leq \lambda_i(\mathbf{A}) + \epsilon n. \quad (1)$$

- Need to sample submatrix with size $\propto \frac{1}{\epsilon^3}$: **sublinear in n**.

Proof Techniques

- Eigendecomposition of **A**: $\mathbf{A} = \mathbf{A}_o + \mathbf{A}_m$.
- $\mathbf{A}_o$: all **"large"** eigenvalues of **A** with $|\lambda_i(\mathbf{A})| \geq \epsilon\sqrt{\delta}n$.
- $\mathbf{A}_m$: all **"small"** eigenvalues of **A** with $\lambda_i(\mathbf{A}) \leq \epsilon\sqrt{\delta}n$.
- $\mathbf{A}_S = \mathbf{A}_{oS} + \mathbf{A}_{mS}$ (after sampling).
- **Eigenvalue Perturbation Theorem:** $|\lambda_i(\mathbf{A}_S) - \lambda_i(\mathbf{A}_{oS})| \leq \|\mathbf{A}_{mS}\|_2$.
- Bound **small eigenvalues** $\|\mathbf{A}_{mS}\|_2$ using known spectral norm bounds from Tropp [2].
- **Intuition: Incoherent eigenvectors of $\mathbf{A}_o$**: By proposition 3.4 of [3] if $\lambda_i(\mathbf{A}) \geq \epsilon n$, $\|x\|_\infty \leq \frac{1}{\epsilon\sqrt{n}}$, ($x$ is the eigenvector associated with $\lambda_i(\mathbf{A})$). Since eigenvectors of $\mathbf{A}_o$ are spread out (**incoherent**), uniform sampling preserves the values approximately.
- Formally, bound **large eigenvalues** $\lambda_i(\mathbf{A}_{oS})$ using an application of **Matrix Bernstein** bound.
- **Connection to leverage score sampling**: Since eigenvectors are **incoherent**, leverage scores of the rows of the matrix of eigenvectors of $\mathbf{A}_o$ are bounded. Thus we can sample using leverage scores to get close spectral approximation.

Lower Bound

Theorem 2 *(General lower bound)*

We need at least $\mathbf{\Omega}(\frac{1}{\epsilon^2})$ samples of any $n \times n$ symmetric matrix to get a $(1+\epsilon)$ factor approximation of the minimum eigenvalue with high probability.

- Generate 2 symmetric $n \times n$ matrices with $0/1$ entries by tossing 2 coins with probability of heads 0.5 and $0.5(1+\epsilon)$.
- Maximum eigenvalue of these matrices follows a **normal distribution** asymptotically (Furedi and Kolmos).
- Need at least $\Omega(\frac{1}{\epsilon^2})$ samples to distinguish between the coins.

Open Questions

- Can sample complexity of upper bound be reduced to $\tilde{O}(1/\epsilon^2)$?

Empirical evaluation

Dataset. We use a synthetic dataset created by uniformly sampling 5000 points from a binary image. We then compute the similarity function, δ, using the following two measures: (a) Sigmoid: $\delta(x,y) = \tanh\left(\frac{xy}{\sigma+1}\right)$, and (b) Thin plane spline (TPS): $\delta(x,y) = \frac{|x-y|^2}{\sigma^2}\log\left(\frac{|x-y|^2}{\sigma^2}\right)$.

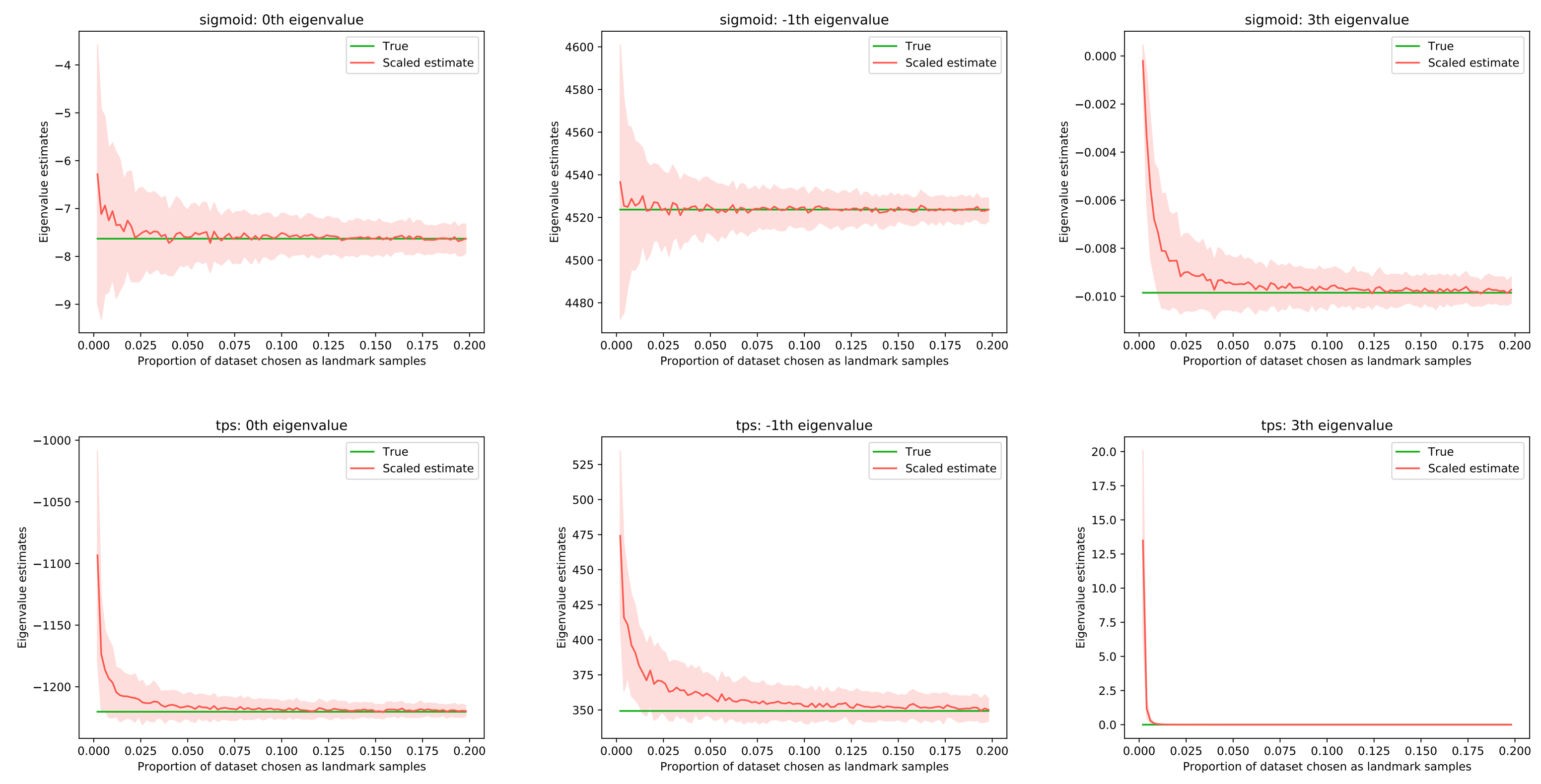

Figure: **Eigenvalue estimates**. Eigenvalues of sigmoid and TPS matrices.

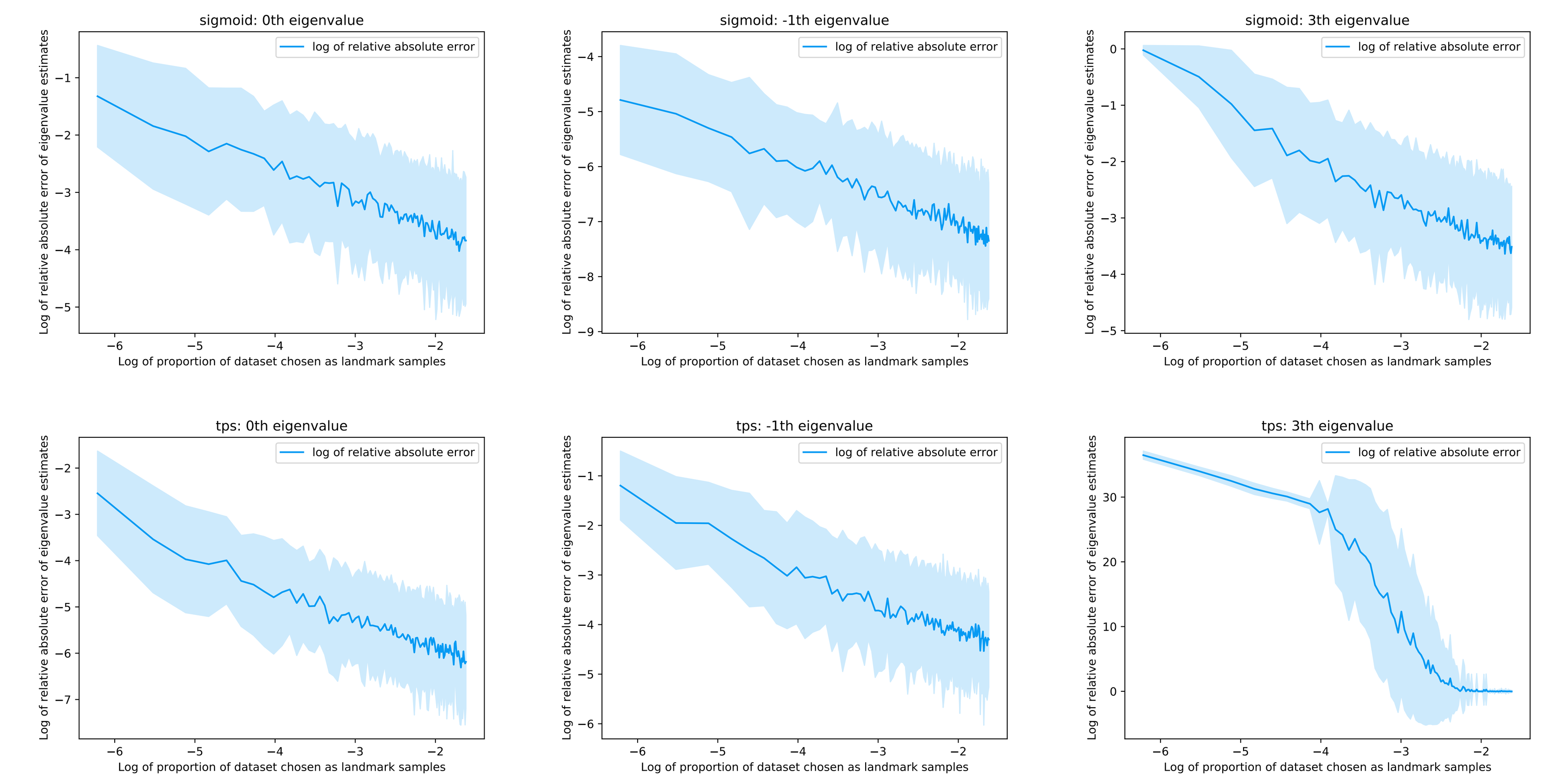

Figure: **Error estimates**. Estimation errors of sigmoid and TPS matrices.

References

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[2] Tropp, J. A. An introduction to matrix concentration inequalities. *arXiv preprint arXiv:1501.01571*, 2015.

[3] Bakshi, A., N. Chepurko, R. Jayaram. Testing positive semi-definiteness via random submatrices. *arXiv preprint arXiv:2005.06441*, 2020.